Philosophy 211
Definitions and Terms used

A note to the reader: There are many different ways of defining some of these terms which in the end turn out equivalent. In all cases the definition I give agrees with the book's definition if it has one. But in reading elsewhere you may come across different definitions for some of these terms.

Throughout, Γ refers to an arbitrary set of sentences such as {P, Q, R} = Γ or {∀xPx, ∀xQx, ∃x∀yRxy} = Γ

φ and ψ refer to arbitrary sentences such as when P ├ Q might be represented as φ ├ ψ or the conjunction φ & ψ might represent any of P&Q, ∀xPx & ∀xQx, or ∃xPx & (Pa v Qb)

=def means that what follows is the definition of what came before (is equal by definition to…) Example: The definition of a tautology is a sentence which is true in every truth value assignment. It is true that all tautologies are theorems, but it is not true *by definition* that all tautologies are theorems. All tautologies are theorems because of the Completeness Theorem.

Proof terms:

Γ ├ φ =def there is a proof of φ (φ is the last line) where all of the premises are members of Γ (there might be more members of Γ than are used in the proof).

Example: {P→Q, P} ├ Q

Γ is proof-inconsistent =def Γ ├ φ and Γ ├ ~φ for some φ.
Γ is proof-consistent =def Γ is not proof-inconsistent.

Examples: {P&Q,Q→R} is proof-consistent. {P, Q, ~Pv~Q} is proof-inconsistent.

φ is a theorem =def ├φ (that is, Γ ├φ when Γ=the empty set which has no members – so φ is provable from no premises at all.)

Example: ├Pv~P

φ is stronger than ψ =def φ ├ψ and it is false that ψ ├φ.
φ is weaker than ψ =def ψ is stronger than φ.
φ is proof-equivalent to ψ =def φ ├ψ and ψ ├φ

Examples: P is stronger than PvQ
Q is weaker than R&Q
~(PvQ) is proof-equivalent to ~P&~Q

Semantic terms:

Γ ╞φ means that φ is a consequence of Γ. It also means that Γ entails φ.

$\Gamma \models \varphi$ =def every interpretation (TVA in SL) that makes every sentence in Γ true also makes φ true. This is equivalent to saying that there is no interpretation (TVA in SL) that makes all the members of Γ true and also makes φ false.

Example: $\{\forall xPx, \forall x(Px \rightarrow Qx)\} \models \forall xQx$

Γ is consistent =def there is an interpretation in which all of the members of Γ are true.
Γ is inconsistent =def Γ is not consistent.
An interpretation which makes all of the members of Γ true is called a Model of Γ.

Examples: $\{\exists xPx, \exists x{\sim}Px\}$ is consistent.
$\{\forall xPx, \forall x{\sim}Px\}$ is inconsistent.
I = U:{a} is a model of $\{\exists x\ x{=}x\}$

φ is a tautology (SL term) =def φ is true in every truth-value assignment
φ is valid (PL term) =def φ is true in every interpretation.
φ is a contradiction =def there is no interpretation in which φ is true.
φ is contingent =def φ is neither valid nor a contradiction. (This is equivalent to saying that there is an interpretation in which φ is true and at least one in which φ is false.)

Examples: $P \vee {\sim}P$ is a tautology
$\exists x(Px \vee {\sim}Px)$ is valid
$\exists xPx$ is contingent
$\exists x(Px \ \& \ {\sim}Px)$ is a contradiction

φ and ψ are independent =def all four of $\{\varphi, \psi\}$, $\{\varphi, {\sim}\psi\}$, $\{{\sim}\varphi, \psi\}$, $\{{\sim}\varphi, {\sim}\psi\}$ are consistent.
φ is independent of Γ =def $\Gamma \cup \varphi$ (Γ added together with φ) and $\Gamma \cup {\sim}\varphi$ are both consistent.
Γ is independent =def there is no $\Gamma_1 \subset \Gamma$ (no subset of Γ) such that $\Gamma_1 \models \varphi$ for some φ in Γ and φ not in Γ_1. In other words, if you take away some sentence φ from Γ then the rest of the sentences don't entail φ.

Examples: $\exists xPx$ and $\exists xQx$ are independent.
$\exists xPx$ is independent of $\{\forall x(Px \rightarrow Qx), \forall x(Qx \rightarrow Rx)\}$
$\{\forall xRxx, \forall x\forall y(Rxy \rightarrow Ryx), \forall x\forall y\forall z((Rxy \& Ryz) \rightarrow Rxz)\}$ is independent.

Metatheorems that relate proofs and semantics:

The Soundness Theorem says that for any Γ and for any φ, if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$. This is equivalent to: if $\Gamma \cup {\sim}\varphi$ is proof-inconsistent then $\Gamma \cup {\sim}\varphi$ is inconsistent. This is equivalent (by contraposition) to if $\Gamma \cup {\sim}\varphi$ is consistent then $\Gamma \cup {\sim}\varphi$ is proof-consistent. Since these are arbitrary this is equivalent to: for any Γ, if Γ is consistent then Γ is proof-consistent.

The Completeness Theorem is the converse of the Soundness Theorem. It says that for any Γ and any φ, if $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

Other major theorems of PL:

Compactness Theorem: Γ is consistent if and only if every finite subset of Γ is consistent.

Church's Theorem: First order logic is undecidable (as long as the language contains at least one two-place predicate). This means that there is no algorithm for correctly answering yes or no to the following question: is $\Gamma \models \varphi$ true or false? (this algorithm would have to work for any Γ, φ) Equivalently, there is no algorithm for answering: is Γ consistent? Is φ a theorem?, etc.